The Relationship between Permutation Groups and Permutation Polytopes

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Abstract

The aim of this paper is to give a new definition of permutation polytope and its relationship to permutation group, the converse from permutation polytope to permutation group is also given according to this relationship, an open conjecture is proved.

1. Introduction

A permutation group is the oldest type of group's representation; it is introduced by Galois, which is generally considered as the start of group theory. Nowadays, permutation group's algorithm is among the best developed parts of computational group theory. The cycle's notion for permutation and the identity permutation which is denoted by e is used. The group of all permutation with n-element denoted by S_n, (Bollobas, Fulton, Katok, Kirwan, & Sarank 2002). A permutation polytope is the convex hull of the permutation matrices of a subgroup of S_n. This polytope is a special class of 0/1-polytope. An example of permutation polytope is the Birkhoff polytope, which is the set of doubly stochastic matrices. This polytope is defined as the convex hull of all permutation matrices. The dimension of a polytope is the dimension of its affine hull. A lot of applications are given in (Shatha & Noor 2015, Shatha 2013, Shatha & Fatema, 2009). This work consists of some basic definitions and some examples of permutation polytope. The permutation group is converted to a permutation polytope is obtained by the definition. Some examples of permutation group are symmetric group S_n, alternating group A_n, dihedral group D_n, Frobenius group F_n and cyclic group C_n. These kinds of permutation group are subgroups of a symmetric group. The convert of permutation groups to permutation polytopes is also given.

2. Basic Concepts

In this section some basic definitions and theorems are given to consolidate our results.

Definition 2.1. (Reff, 2007):

All matrices are obtained based on $n \times n$ identity matrix, this is through permuting the rows the resulting matrices are called permutation matrices and denoted by $M(G)$. So the number of permutation matrices is $n!$.

Example 2.1.

An example of permutation matrices is given as follows

Let n=2 with identity matrix equal $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the by permuting of this matrix, we obtained $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Definition 2.2. (Burggraf, 2011):

Let A be a finite set {1, 2,…, n}, a permutation group is defined as a group whose element are permutations of A.

A permutation polytope is defined as the convex hull of a group or subgroup of permutation matrices, (Baumeister, Haase & Paffenholz, 2009).

Theorem 2.2. (Judson,1997):

Let G be a group of finite order n, and H a subgroup of G. The order of H divides the order of G.

Example 2.2.

Let S_4 be a group of order 24 ,its subgroups are given in table 1.

Table 1 Subgroups of S_4.

Order	$H \leq S_4$
1	{e}
2	C_2
3	C_3
4	C_4
6	S_3
8	D_4
12	A_4
24	S_4

3. The Relation between Permutation Groups and Permutation Polytopes

In this section, the relationship between the groups of permutation and permutation polytopes is discussed.

3.1. Permutation Polytopes from Permutation Groups

Let G be a set of all permutations of n element such that [n]={1,2,...,n}. For every permutation σ in G, we associated a matrix m^{σ} for each $\sigma \in G$, where m^{σ} is a permutation matrix, such that $m^{\sigma} \in R^{n\times n}$ is defined as (1), (Baumeister, Haase, Nill & Paffenholz, 2012).

$$m_{ij} = \begin{cases} 1 & \text{if} \quad \sigma(i) = j \\ 0 & \text{otherwise} \end{cases} \quad \ldots\ldots\ldots\ldots (1)$$

Then,

$$m^{\sigma} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{1n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

Example 3.1.1. (Reff, 2007):

Let n=3, in order to convert a permutation group to a permutation polytope, G is defined as,

G={e,(123),(132),(12),(13),(23)}.

Take all elements of a permutation group $\sigma \in G$ and convert it to permutation matrix that is

$$m^{(\sigma)} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

if σ= (e) =(1)(2)(3), then

$\sigma(1) = 1 \qquad m_{11} = 1$
$\sigma(2) = 2 \qquad m_{22} = 1$
$\sigma(3) = 3 \qquad m_{33} = 1$

We obtain the permutation matrix $m^{(e)}$ associate to the permutation element e such that

$$m^{(e)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If σ= (123), then

$\sigma(1) = 2 \qquad m_{12} = 1$

$\sigma(2) = 3 \quad m_{23} = 1$
$\sigma(3) = 1 \quad m_{31} = 1$

We obtain the permutation matrix $m^{(123)}$ which is associated to the permutation element (123) that is

$$m^{(123)}= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

In a similar way, we obtain the permutation matrices, which are;

$$m^{(132)}= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ m^{(12)}= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$m^{(13)}= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ m^{(23)}= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hence,

$$M(G)=\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

Permutation polytope, P(G)=conv(M(G)).

3.2. Permutation Groups Obtained from Permutation Polytopes (Burggra, Loera, & Omer, 2013, Mcmahan, 2003):

Let P(G) be the permutation polytope, which is equal to conv(M(G)), for all $A_i \in$ M(G) implies to elements of a permutation group, $i = 1,2,\dots,n$.

Definition 3.2.1.

Let g be one element of the permutation group G, let A be any matrix in the set of permutation matrices, then g is given by (2), such that

$$\mathbf{g}=F(\mathbf{A}) = \begin{cases} i & \text{if } a_{ij} = 1 \text{ and } i = j \\ j & \text{if } a_{ij} = 1 \text{ and } i \neq j \\ \text{neglected} & \text{if } a_{ij} = 0 \end{cases} \quad \dots\dots\dots\dots\dots(2)$$

Example(3.2.1):

To convert permutation matrix to a permutation group. The following steps is used

First:

Convert permutation polytope to a permutation group where permutation polytope p(G)=conv(M(G)) and

$$M(G)=\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

Second:

Taking any matrix A_1 from the permutation group , where g be any element of the permutation group, then

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_1=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3) = e$$

$A_2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}=(123)$

$A_3=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}=(132)$,

In a similar way, we compute

$\sigma_4=(12)(3)$, $\sigma_5=(13)(2)$ and $\sigma_6=(1)(23)$

Therefore , the permutation group={e,(123),(132),(12),(13),(23)}

4. Symmetric group with the permutation polytopes

In this section the symmetric group is discussing and we will convert it to permutation polytope.

Definition 4.1. (Judson,1997):

Symmetric group is defined as a group of all permutations of A, where A is the finite set {1,2,...,n}, the group is said to be a symmetric group on n and denoted by S_n, it has $n!$ elements.

Remark 4.1. (Burggra, Loera, & Omer, 2013):

1- The Birkhoff polytope $B_n= P(S_n)=\text{conv}(M(S_n))$.

2-The polytope that associate the symmetric group have same dimension of the Birkhoff polytope B_n, which is $(n-1)^2$.

Example 4.1

Let n=4, take a symmetric group S_4, we want to find the permutation polytopes that are relates to S_4.

Solution:

For n=4, S_4 is defined as:

S_4={e,(1234),(1243),(1324),(1342),(1423),(1432),(12)(34),(13)(24),
(14)(23),(234),(243),(134),(143),(124),(142),(123),(132),(12),(13), (14),(23),(24),(34)}

By converting the symmetric group S_4 to permutation matrices we obtain

$$M(G)=\left\{\begin{array}{l}
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix},\\
\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\\
\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix},\\
\begin{pmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{pmatrix},\\
\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix},\\
\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}
\end{array}\right\}$$

$G_1 \leq M(G)$,[11]

$$G_1=\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right\}$$

$P(G_1)$=conv(G_1).

The number of vertices for this polytope $P(G_1)$ is four, which are:

(1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1),(0,1,0,0,1,0,0,0,0,0,1,0,0,0,0,1),(1,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0) and (0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0) are defined in R^{16}.The permutation polytope $P(G_1)$is square in two dimensional embedding in the sixteen dimensional space.

5. Alternating Group with the Permutation Polytopes

In this section, alternating group is discussed; it is converted to a permutation polytope.

Theorem 5.1. (Burggraf,2011):

For $n \geq 4$, $\dim(P(A_n)) = \dim(P(S_n)) = (n - 1)^2$.

Example 5.2. Baumeister, Haase, Nill, & Paffenholz, 2014):

For $n = 3$, an alternating group with its permutation polytope is found as follow, A_3={e,(123),(132)}, their elements are $n!/2$, which is equal to three. Since all permutations of A_3 is even permutation, therefore, by converting permutation groups to permutation matrices, we obtain;

$$M(g)=\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\right\}$$

$P(A_3)$=conv(M(g))

The vertices for this polytope are (1,0,0,0,1,0,0,0,1), (0,1,0,0,0,1,1,0,0) and (0,0,1,1,0,0,0,1,0). All vertices in R^9. The permutation polytope $P(A_3)$ is a triangle in dimension two, all embedded in nine dimensional space.

6. Dihedral Group with the Permutation Polytopes

In this section a dihedral group is given, it is also converted to a permutation polytopes.

Definition 6.1.(Baumeister, Haase, Nill & Paffenholz, 2012):

a polygon with n vertices is called n-gon, where polygon is a polytope in two dimensions.

Remark 6.1.(Judson,1997):

The n^{th} dihedral group is the group denoted by D_n of symmetries of the regular n-gon and the dihedral group of order 2n.

6.1. Some properties of a dihedral group, (Collins & Perkinson 2011):

1-Dihedral group is a subgroup of symmetric group $D_n \leq S_n$, $n \geq 3$
2-The permutation polytope denoted by $P(D_n)$=conv($M(g)$), $\forall g \in D_n$.
3-The dimension of $P(D_n)$ is at most 2n.

6.2. A Method for Finding the Elements of a Dihedral Group, (Collins & Perkinson 2011):

1-Take the rotation ρforn-gon by 360/n degrees
2-The reflection τ is define as

$$\tau = \begin{cases} (2,n)(3,n-1)\ldots(\frac{n+1}{2},\frac{n+3}{2}) & \text{If n odd} \\ (2,n)(3,n-1)\ldots(\frac{n}{2},\frac{n}{2}+2) & if\ n\ even \end{cases}$$

3-The Dihedral group D_n has an element, which are

$\rho^0, \rho^1, \rho^2, \ldots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \ldots, \tau\rho^{n-1}$

Figure 1 and 2 represent the operation of finding the elements of a dihedral group, (Ziegler, 1995).

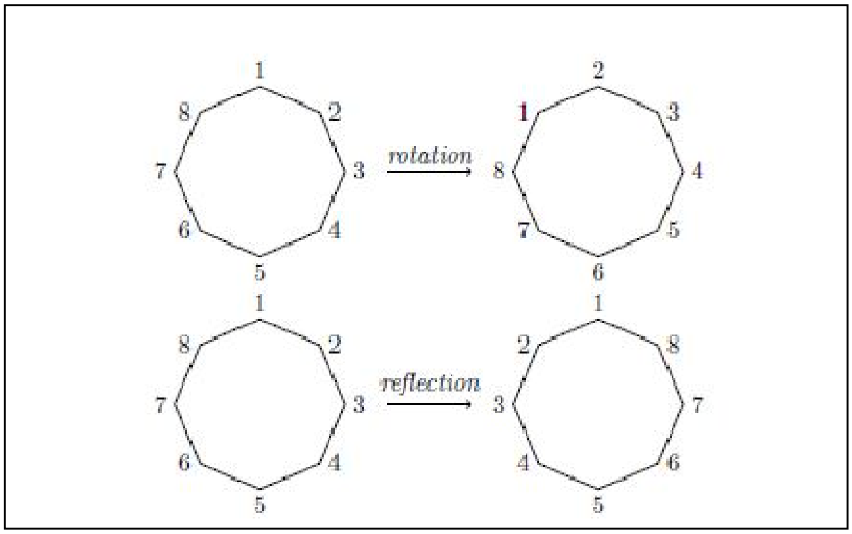

Figure 1: Rotation and reflection of regular n-gon.

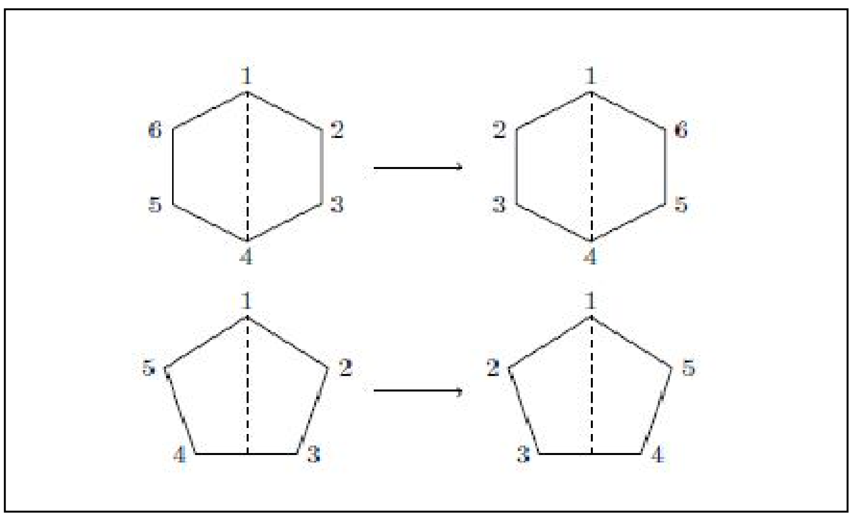

Figure 2: Type of reflection of a regular n-gon.

Lemma 6.1.(Mcmahan, 2003):

The dimension of the polytope $P(D_n)$ is (2n-2) if n is odd and (2n-3) if n is even.

Example 6.1.

For n=4 the dihedral group is found, with the permutation polytopes. Since n=4, the permutation group corresponding to 4-gon, which are squares of four vertices.

Solution

Using the notion given in Sec 5.2, we have the degree of rotation 360/4=90 therefore,

$\rho^0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = e$, $\rho^1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234)$,

$\rho^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$ and $\rho^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1432)$

Since n=4 is even then,

τ =(2,n)(3,n-1)…(n/2,(n/2)+2), $\tau = (2,4)$

Therefore, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (24)$

$\tau\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$

$\tau\rho^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (13)$

$\tau\rho^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$

There are, the dihedral group for n=4 is,

$D_4 = \{\rho^0, \rho^1, \rho^2, \rho^3, \tau, \tau\rho, \tau\rho^2, \tau\rho^3\}$

D_4 ={e,(1234),(13)(24),(1432),(24),(12)(34),(13),(14)(23)}

Now, by converts the permutation group to a permutation polytopes, we get

$$M(g) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$P(D_n)$=conv($M(g)$)

The vertices for this polytope is eight

(1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1),(0,1,0,0,0,0,1,0,0,0,0,1,1,0,0,0),(0,0,1,0,0,0,0,1,1,0,0,0,0,1,0,0),(0,0,0,1,1,0,0,0,0,1 ,0,0,0,0,1,0),(1,0,0,0,0,0,0,1,0,0,1,0,0,1,0,0),(0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0),(0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,1) and (0,0,0,1,0,0,1,0,0,1,0,0,1,0,0,0) these vertices in R^{16}

The permutation polytope $P(D_n)$ in five dimensional embedding in sixteen dimensional space.

7. Frobenius Group with the Permutation Polytopes:

In this section we discuss Frobenius group and we will convert it to a permutation polytope.

Definition 7.1. (Whitehead, 1988)

Frobenius group F is a finite group given by a permutation group, we have $F = NH$, where N is a group called Frobenius kernel, and H is a group called Frobenius complement, the Frobenius group is denoted by F_n, where n={1,2,…}.

Example 7.1.

An example of a Frobenius group which is the most familiar is a collection of odd dihedral groups where n=3. To find the Frobenius group and the permutation polytope, we have:

Solution

The degree of rotation 360/3=120,

$\rho^0 = \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = e, \rho^1 = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} = (123)$ and

$$\rho^2 = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = (132)$$

Since n is odd, then

τ =(2, n) (3, n-1)…((n+1)/2, (n+3/2)+2)

$\tau = (2,3)$

$\tau = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = (23)$

$\tau\rho = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} o \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} = (12)$

$\tau\rho^2 = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} o \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = (13)$

With Frobenius complement $H = \langle\tau\rangle$

$\tau^0 = e$, $\tau^1 = (23)$, $\tau^2 = e$

H={e,(23)}

Also, with Frobenius kernel N=$\langle\rho\rangle$

$\rho^0 = e$, $\rho^1 = (123)$, $\rho^2 = (132)$

N={e,(123),(132)}

Such that N∩H=e

F_3={e,(123),(132),(23),(12),(13)}

By converting permutation group to a permutation matrices we obtain,

$$M(G)=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}\right\}$$

P(F_3)=conv(M(G))

The vertices for this polytope P(F_3) are

(1,0,0,0,1,0,0,0,1),(0,1,0,0,0,1,1,0,0),(0,0,1,1,0,0,0,1,0),(1,0,0,0,0,1,0,1,0),(0,1,0,1,0,0,0,0,1)and (0,0,1,0,1,0,1,0,0). All vertices in R^9. The permutation polytope P(F_3) in dimension three, all embedded in nine dimensional space.

8. An Open Conjecture with Proof

In this section an open conjecture about permutation group and permutation polytope is given.

Definition 8.1. (Ziegler,1995):

Face poset is defined as a poset of cells of p order by inclusion, where face poset is denoted by $\mathcal{X}(p)$.

Definition 8.2.(Baumeister, Haase, Nill & Paffenholzm, 2009):

An equivalence of the face lattice as a posets of polytopes lattice is said to be combinatorial equivalent and is denoted by $P \approx Q$, where P and Q are two polytopes.

Remark 8.1.(Baumeister, Haase, Nill & Paffenholz, 2012):

Two polytopes are combinatorially equivalent in dimensions two, if and only if, they have the same number of vertices.

Remark 8.2.

If two polytopes are combinatorially equivalent, then they have the same number of vertices and same dimension. Now we prove an open conjecture for (Baumeister, Haase, Nill & Paffenholzm, 2009).

Theorem 8.1.

Let P be a d-dimensional permutation polytope, then there exist a permutation group $G \leq S_{2d}$, such that, P(G) is combinatorial equivalent to P.

Proof

Let P be a d-dimensional permutation polytope, P(G) is combinatorially equivalent to P, a permutation polytope of dimension d is defined as the convex hull of a group or subgroup of permutation matrices.Therefore, the number of vertices of P is equal to 2d or less, that is a subgroup of a permutation group G.

In dimension two, two polytopes are combinatorially equivalent if and only if they have the same number of vertices therefore, from, the definition of combinatorially equivalent, the faces lattice of P and P(G) are isomorphic, that is $G \leq S_{2d}$

References

Baumeister, B., Haase, C. , Nill, B. & Paffenholz, A. (2014). Polytopes Associated to Dihedral Groups", ARS MathmaticContemporanea,7, 30-38.

Baumeister, B., Haase, C., Nill, B., & Paffenholz, A. (2012).Permutation Polytopes of Cyclic Groups, Discrete Mathematics and Theoretical Computer Science Proce.AR, 423-434.

Baumeister, B., Haase, C., Nill B. and Paffenholz, A.(2009).On permutation polytopes , Advances in mathematics,222, 431-452.

Burggraf, K., Loera, J.D. & Omer, M. (2013). On volumes of permutation polytopes, Fields Institute Communications, Discrete geometry and optimization,69, 55-77.

Burggraf, k. (2011).Characterizing Automorphism and Permutation Polytopes, B.Sc.Thesis of the University of California.

Bollobas, B., Fulton, W., Katok, A., Kirwan, F. & Sarank, P. (2002).152 Permutation Groups Algorithms, Akos Seress.

Collins, J. & Perkinson, D. (2011). Frobenius Polytopes, arXiv:1102.0988v1.

Judson, T. W. (1997).Abstract algebra theory and applications, published by Thomas W. Judson.

Mcmahan, P. (2003).Wreath-Product polytopes, B.Sc.thesis of Reed College.

Minian, E.G.(2012).Some remarks on Morse Theory for Posets, homological Morse theory and Finite Manifolds, Topology and its applications,159, 3860-2869.

Reff,N. (2007). A generalization of the Birkhoff-von Neumnn Theorem, M.Sc.Thesis of Rochester Institute of Technology.

Shatha A.S. & Noor, R. Sh. (2015). The Maximal Number of Vertices of the Reflexive Polytope, Eng & Tec Journal,33(4), part(B),733-744.

Shatha A.S. (2013). Computing the Number of Integral Points in 4-Dimensional Ball, Scientia Magna, 9(1), 21-26.

Shatha A.S. and Fatema,A.S. (2009). Ehrhart Polynomials of a Cyclic Polytopes, Eng & Tec Journal, 27(14), 2624-2631.

Whitehead, C. (1988).Guide to Abstract Algebra, Carol Whitehead.

Ziegler,G.M. (1995). Lectures on polytopes, Springer-Verlag New York, Inc.